Pergamon

Int. J. Solids Structures Vol. 33, No. 20–22, pp. 3327–3342, 1996

0020–7683/96 $15.00 + .00

PII: S0020–7683(96)00015–7

FROM DAMAGE TO FRACTURE MECHANICS AND CONVERSELY: A COMBINED APPROACH

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Abstract—Fracture mechanics and damage mechanics are two correlated theories. In some instances, e.g., for large specimens, crack propagation may be viewed equivalently as a sudden localization of damage. Relationships based on thermodynamic considerations between the two theories are presented in this paper. They lead to the definition of the equivalent crack concept, in passing from a damage zone to a fracture problem and, conversely, a damage zone is determined which is equivalent to a crack. Different possible applications are presented showing that, for the same problem, the two concepts can be used depending on the situation. Furthermore a solution to calculate fracture energy for large specimens, when damage parameters deduced from classical tests are known, is proposed to illustrate the capability of these equivalences. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

When quasi-brittle materials such as cementitious composites and concrete are subjected to external mechanical loads they exhibit a non linear response which is mainly due to damage and micro cracking. Experimental observations show that main (macro) cracks develop on the surface of concrete specimens. What is analysed indirectly using specific systems (acoustic emissions, ultra-sonic waves,...) is also diffuse micro cracking located around the main (macro) crack. These micro cracks form the so-called process zone.

There are two main categories of models which describe this failure process: fracture mechanics and continuous damage mechanics.

- Fracture mechanics is well suited to describe the separation due to the decohesion of two parts of the continuum (Kaplan, 1961, Mazars, 1977). It can be applied once a crack has been initiated, or assuming that there are initial flaws of known sizes and known locations in the continuum.
- Damage mechanics, which includes smeared (or distributed) crack models (Bazant and Oh, 1983, de Borst and Nauta, 1985, Mazars and Pijaudier-Cabot, 1989), describes the local effects of micro cracking, that is, the evolution of the mechanical properties of the continuum as micro cracking develops: elastic stiffness degradation, induced anisotropy, anelastic strains...

When the location of an expected crack and the direction along which it propagates are unknown, fracture mechanics can hardly be used because the critical flaw from which cracking initiates needs to be determined first. On the contrary, damage mechanics offers the essential advantage to predict the location of this critical flaw. Conversely, linear or non linear criteria for crack propagation exist in the literature on fracture mechanics while the relationships between these criteria and the evolution of damage in the continuum sense are difficult to define. The bridge between fracture and damage mechanics can be considered to be the situation where damage is equal to one at a material point, or in a small region defining the size of an initial flaw in the theories of fracture. In most instances, this situation corresponds to the localization of strains and damage due to strain softening. Strain softening yields, however, several problems which need to be solved in order to bridge the two theories, and to insure the objectivity of the numerical modelling and the well posedness of boundary value problems at the onset of localization. More importantly, the major

obstacle which is faced when fracture and classical (local) damage theories are compared is that failure predicted with classical damage models occurs without dissipation of energy (Bazant, 1976). Damage models predict that the density of dissipated energy is finite at each material point but since damage localizes in a region of zero volume, the total amount of energy dissipated to form a crack, viewed as a line along which damage is equal to one at each material point, vanishes. Most criteria for crack propagation are based on the quantity of energy which must be released in order to propagate a crack. Hence, there is a major inconsistency between classical damage theories which predict that this quantity of energy is zero and fracture mechanics which assumes that the same quantity is finite, non zero.

This paradox has been recognized in the literature twenty years ago. Bazant pointed out this discrepancy on the simple problem of a bar and a beam made of a strain softening material (1976) and proposed further to implement non local models to circumvent the problem and predict size effects (see Bazant, 1986). Simo (1989) also identified this paradox and showed that most continuum models aimed at solving it should enforce the fact that the amount of energy dissipation at failure is a finite, non zero quantity. Along this line, Simo *et al.* (1993) proposed the modelling of strong discontinuities, cracks in the continuum sense, in such a way that the energy dissipated per unit surface of discontinuity would remain finite. Hence this attempt at closing the gap between continuous damage theories and fracture mechanics was successful in the sense that the two theories were not only consistent from the energy point of view but also from the view point of the description of the field variables, strain and other internal variables aimed at modelling the crack and the fracture process zone.

Progressive cracking can also be introduced in fracture mechanics as a cohesive zone, i.e., a pressure distribution of the crack faces near the tip, which is a way to concentrate the volume distribution of micro cracks into a surface (Elfgren, 1989). This approach results in the fictitious crack model. The relationship between fracture and damage is, however, merely obtained on the basis of an energy equivalence : the work of the pressure distribution introduced near the crack tip is equivalent to the energy dissipated during fracture. Since the fracture process zone is collapsed on to a line, the distribution of damage in this zone cannot be obtained and this approach cannot be used for modelling diffuse damage.

Non local damage models (Pijaudier-Cabot and Bazant, 1987) or gradient dependent models (de Borst *et al.*, 1992) can also be regarded as a consistent way of closing the gap between these two theories. The key idea of non local damage models is to assume that the condition of growth of damage is non local, i.e., that it depends at each material point on a weighted average of the strains in a neighbourhood. This neighbourhood is scaled by an internal length parameter related to the size of the heterogeneities (Bazant and Pijaudier-Cabot, 1989). The distribution of damage in the process zone is obtained as a result of the computation. Even though micro cracking is modelled in a crude fashion, i.e., by a scalar defining the degradation of the Young's modulus of the material which remains isotropic in spite of the preferential orientation of the micro cracks, several numerical results show that the paradox is avoided as far as the energy dissipation at failure is concerned. Many experimental results on the size effect of tensile fracture can be reproduced and in the limit of specimens of infinite sizes, linear fracture mechanics is recovered (Mazars *et al.*, 1991). Hence, non local damage modelling can capture diffuse and localised cracking in the same framework, without any need for assuming initial flaws in the structure as opposed to fracture mechanics. Spurious mesh dependency due to damage localization is also avoided (Saouridis and Mazars, 1992). Due to the simple definition of the damage variable, the applicability of this model is certainly very limited, e.g., to tensile fracture, but the same principles have presided over the development of more general damage models, such as the microplane approach which incorporates damage induced anisotropy due to the preferential orientation of micro cracking (see, e.g., Bazant and Ozbolt, 1990, Carol *et al.*, 1991).

One striking fact is that non local damage and fracture treat the same kind of problems and lead, in the cases of prediction of fracture mentioned above, to similar results. From these considerations, the purpose of this paper is to provide a view on the possible analytical connections between non local damage and fracture mechanics. Such a connection already

exists for gradient models (de Borst *et al.*, 1992) but not for damage models. The major difficulty in this problem is the relationship between the material parameters in the continuum model and the critical parameters in the theories of fracture. The objective of this exercise is to offer the possibility of passing from one theory to the other during one calculation or to obtain, from one theory, information to use in the other. This information is mostly related to an equivalence of energy dissipated according to the two theories. The usefulness of setting this equivalence between the two theories is, at a given modelling point, to be able to choose the most efficient or the most simple approach. Even if non local damage modelling encompasses both descriptions of diffuse and localized damage (e.g., crack propagation), it may be interesting to switch during the computation to fracture types of approaches because they are computationally more efficient. This is the case of large size structures where the finite element discretisation of the damage zone would require a prohibitive amount of nodal unknowns. Conversely, the determination of the model parameters in a damage law should benefit from the fracture tests proposed in the literature or at least be consistent with them. This piece of information cannot be neglected whenever the determination of model parameters involved in continuum constitutive relations needs to be performed. Finally, there are practical situations where a structure exhibits an initial crack. Since non local damage modelling is able to describe the behaviour of initially uncracked structures without introducing an initial flaw, we may want to use the same approach to predict the response of an initially cracked structure. Thus, the equivalence between an existing crack, its process zone and a distribution of damage needs to be obtained.

This paper is organized as follows: fracture and damage theories are recalled in Section 2. Starting from the same framework which is thermodynamics of irreversible processes, an energetic equivalence between the two descriptions is proposed. Section 3 deals more specifically with the bridge between damage and fracture. The energy consumption during crack propagation, modelled with damage mechanics, is computed. Applications are presented in Section 4. We start by presenting an example of combined damage and fracture calculation on a concrete specimen. Then, the results of Section 3 are implemented in order to compare the fracture energy according to the damage model with experiments and linear elastic fracture mechanics. Finally, an example of computation of an initially cracked specimen with the continuum damage approach is discussed.

2. THEORETICAL CONCEPTS

In order to be at the same time simple and demonstrative, the presentation which follows assumes that the behaviours concerned are linear elasticity coupled with damage or fracture and that the evolutions are at fixed temperature.

2.1. *Thermodynamic basis*

A unified manner to present damage and fracture mechanics is thermodynamics (see, for this, Lemaitre and Chaboche, 1990). Thermodynamics deals here with energetic considerations, from which it is easy to relate local damage variables and global fracture variables. These considerations start with the assumption of a specific form of the free (reversible) energy stored in the material during straining. Let us emphasize that the paper deals with the simplest possible forms of such energies.

The state equations are deduced from the free energy defined as:

$$\Psi = U - TS \quad (U, \text{internal energy}; T, \text{temperature}; S, \text{entropy}). \tag{1}$$

For the elementary volume at a given state of damage:

$$u = \tfrac{1}{2}\Lambda^{D}_{ijkl}\varepsilon_{ij}\varepsilon_{kl}. \tag{2}$$

For the overall body, damaged or partially cracked:

$$U = \tfrac{1}{2} K q^2. \tag{3}$$

Λ^D_{ijkl} is the local stiffness matrix at a given stage of damage; ε_{ij} is the local strain component. A load denoted as Q is applied to the structure, q is the corresponding displacement and K is the global stiffness. Assuming linear elasticity and that damage is isotropic, the relationship between Λ^D_{ijkl} and the initial stiffness of the undamaged material is:

$$\Lambda^D_{ijkl} = \Lambda_{ijkl}(1-D) \tag{4}$$

where Λ_{ijkl} is the stiffness matrix for the virgin material, with constant values depending on the Young's modulus and the Poisson ratio for an isotropic material which is linear elastic. At uniform and constant temperature the state laws give:

—for the damaged material (see Lemaitre, 1992), it comes from (1), (2) and (4):

$$\sigma_{ij} = \frac{\partial \Psi}{\partial \varepsilon_{ij}} = \Lambda_{ijkl}(1-D)\varepsilon_{kl} \tag{5}$$

$$Y = \frac{\partial \Psi}{\partial D} = -\tfrac{1}{2}\Lambda_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \tag{6}$$

where Y is the damage energy release rate,

—for the cracked structure (A is the actual area of the crack) (1) and (3) give (Lemaitre and Chaboche, 1990):

$$Q = \frac{\partial \Psi}{\partial q} = Kq \tag{7}$$

$$G = \frac{\partial \Psi}{\partial A} = \tfrac{1}{2} q^2 \frac{\partial K}{\partial A}. \tag{8}$$

Apart from the sign, G is the fracture energy release rate.

The first and second principles of thermodynamics are completely satisfied if the Clausius-Duhem inequality is also verified (Lemaitre and Chaboche, 1990). For the two considered cases, we obtain:

$$-Y\dot{D} \geqslant 0 \rightarrow \tfrac{1}{2}\Lambda_{ijkl}\varepsilon_{ij}\varepsilon_{kl} D \geqslant 0 \tag{9}$$

$$-G\dot{A} \geqslant 0 \rightarrow \tfrac{1}{2} q^2 \left(-\frac{\partial K}{\partial A}\right)\dot{A} \geqslant 0. \tag{10}$$

Since $(-Y)$ is a quadratic function and K decreases when A increases, these equations imply that $\dot{D} \geqslant 0$, and $\dot{A} \geqslant 0$, showing that irreversibilities correspond to micro or macro-cracking propagation.

Assuming a monotonic loading, the equations of evolution used in the present paper are:

—for the damage model

$$\begin{aligned} &g = \tilde{\varepsilon} - \varepsilon_c \\ &g < 0 \Rightarrow \dot{D} = 0 \\ &g = 0 \Rightarrow \dot{D} > 0 \end{aligned} \tag{11}$$

where g is the loading function. $g = 0$ provides the limit of the reversible (elastic) domain,

and ε_c is the present current value for $\bar{\varepsilon}$, that is, for each material point, the maximum between $\bar{\varepsilon}$ ever reached during the loading history and a threshold ε_{D0}.

—for the fracture model

$$\begin{aligned} F &= |G| - G_C \\ F < 0 &\Rightarrow \dot{A} = 0 \\ F = 0 &\Rightarrow \dot{A} > 0 \end{aligned} \tag{12}$$

where $F = 0$ defines the reversible domain (no propagation of crack) and G_C the present critical value for $|G|$.

The damage model used in the computations hereafter (Mazars, 1986) is based on the following, integrated, equation of evolution of damage:

$$D = f(\bar{\varepsilon}, \varepsilon_{D0}, A_t, B_t) = 1 - \frac{\varepsilon_{D0}(1 - A_t)}{\bar{\varepsilon}} - \frac{A_t}{\exp[B_t(\varepsilon - \varepsilon_{D0})]} \tag{13}$$

A_t, B_t are constants, and $\bar{\varepsilon}$ is the non local value of the "equivalent tensile strain" defined as:

$$\tilde{\varepsilon} = \sqrt{\sum_i \langle \varepsilon_i \rangle_+^2}, \quad (\langle \varepsilon_i \rangle_+ \text{ is positive part of the principal strain } \varepsilon_i). \tag{14}$$

$\bar{\varepsilon}$ is the average of $\tilde{\varepsilon}$ over the representative volume surrounding each point x in the material. It is the variable that controls the growth of damage:

$$\bar{\varepsilon}(x) = \frac{1}{V_r(x)} \int_v \tilde{\varepsilon}(s) \alpha(s - x) \, \mathrm{d}s \tag{15}$$

V is the volume of the structure, and $\alpha(s - x)$ is a weight function:

$$V_r(x) = \int_v \alpha(s - x) \, \mathrm{d}s, \quad \alpha(s - x) = H(s - x) \exp\left(- \frac{4|s - x|^2}{l_c^2} \right) \tag{16}$$

where H is the function, equal to 0 if $|s - x| > l_c/2$, equal to 1 if $|s - x| \leqslant l_c/2$. This function operates a truncature in the calculation of the integrals. It simplifies the computation of the non local variable during finite element calculations because the domain of influence around each gauss point is limited.

l_c is the internal length of the non local continuum. It is proportional to the smallest size of the damage localization zone. This length was assessed experimentally on the basis of an energy equivalence between a specimen where damage is constrained to remain diffuse and another one in which damage localizes in order to obtain a single crack (Bazant and Pijaudier-Cabot, 1989). The value $l_c \approx 3d_a$, in which d_a is the maximum size of the aggregate in concrete, can be considered to be a correct approximate for this internal length which is difficult to measure directly (see Mazars and Pijaudier-Cabot (1989), Saouridis and Mazars (1992) for more details).

2.2. *Energetic equivalence*

Considering the similarity of the two approaches, it seems natural to go from one concept to the other. One possible solution is to transform a given damage zone into an equivalent crack or conversely. This equivalence must be thermodynamically acceptable, which means that during the evolution:

$$\int_V -Y\dot{D}\,\mathrm{d}V = -G\dot{A} \tag{17}$$

where V is the overall volume of the structure.

3. BRIDGE BETWEEN FRACTURE AND DAMAGE

There are two ways of defining the connection between damage and fracture theories:

(i) From damage to fracture: the problem consists of a transformation of a given damage zone into an equivalent crack.
(ii) From fracture to damage: this necessitates the transformation of a crack into a volume distribution of damage on a "band" around this crack.

3.1. *From damage to fracture*

Considering the case of LEFM, the critical condition of crack propagation coming from (12) is $-G = G_c$. Then eqn (17) gives the equivalent progression $\mathrm{d}A_e$ of a crack to a given evolution $\mathrm{d}D(x)$ of damage, at point x (Mazars, 1986):

$$\mathrm{d}A_e = \frac{\int_V -Y\,\mathrm{d}D(x)\,\mathrm{d}x}{G_c} \tag{18}$$

If we consider now the total evolution, $0 \to D(x)$ at point x, the *equivalent crack* is:

$$A_e = \frac{\int_V \int_0^{D(x)} -Y\,\mathrm{d}D\,\mathrm{d}x}{G_c} \tag{19}$$

where A_e is the crack, the progression of which consumed the same energy than the energy consumed during the progression of damage in the structure.

3.2. *From fracture to damage*

In the general case for specimens of finite dimensions, the energy consumed during the crack process is G_f, the fracture energy, generally deduced from the area under the load-displacement curve. The simulation of this process is possible using two kinds of modelling:

—the classical one is the use of non linear fracture mechanics, in which the description of non linearities are included in the behaviour of the process zone which is collapsed onto a line [see, e.g., the fictitious crack model from Hillerborg (1976)],

—the other uses mechanics of continuum through a non local damage modelling able to describe the overall process: the cracked zone ($D = 1$) and, ahead and around, the process zone ($0 < D < 1$).

Planas and co-workers (1993) have derived the relationship between non local models for concrete and the fictitious crack model. In these approaches the fracture energy and the softening behaviour are considered as material properties and the link between these characteristics can be derived explicitly. This result shows that with strain softening and a non local model, a fracture energy is implicitly introduced in the calculation. Since this energy is independent from the boundary conditions of the problem, it can be interpreted as the energy necessary to create a crack, that is a line where damage is equal to one and the corresponding process zone, in an infinite body subjected to tension. But the main problem to solve is the strain and damage distribution equivalent to a crack and its process zone, which is generally obtained from finite element calculations. An analytical approximation to these distributions is presented in the following.

Consider a crack as shown in Fig. 1. The crack is replaced by an equivalent damage zone whose mathematical expression is given in a local coordinate system following the crack path. In the following, material points whose location is denoted as x will be more specifically denoted according to this coordinate system $x = (x_1, x_2)$ whenever needed. We restrict the analysis to the mode I fracture energy for an infinite specimen, and we consider that the local state of stress is *uniaxial* tension.

The constitutive relations given in eqns (5, 11–15) are non-linear. Thus we look first at the incremental growth of damage for linearised constitutive relations, starting from an initial state of strain and stress. The rate constitutive relations are:

$$\dot{\sigma}_{ij}(x) = (1-D^0)\,\Lambda_{ijkl}\dot{\varepsilon}_{kl} - \frac{\Lambda_{ijkl}\varepsilon^0_{kl}}{V_r(x)}\left.\frac{\partial f}{\partial \bar{\varepsilon}}\right|_{\bar{\varepsilon}^0}\int_v \alpha(s-x)\dot{\bar{\varepsilon}}(s)\,\mathrm{d}s \tag{20}$$

where $(\varepsilon^0, \bar{\varepsilon}^0, D^0)$ is the initial, homogeneous state of deformation and damage about which the rate constitutive relations are expressed. Assuming that the initial state is in equilibrium, the condition of equilibrium for small perturbations about this state is:

$$\mathrm{div}(\dot{\sigma}) = 0 \tag{21}$$

where body forces are assumed to remain constant during the loading history. Analyses of localization and bifurcation in non local continuum (Pijaudier-Cabot and Benallal, 1993) have shown that the solution of this equation is harmonic. In the local co-ordinate system of the crack (Fig. 1), its expression is:

$$\dot{u}(x_1, x_2) = \mathbf{A}\exp(-i\omega(x_2)) \tag{22}$$

where ω is the angular frequency of the solution and $\mathbf{A}$ is a vector of unknown components. Note that this solution does not depend on the co-ordinate x_1, which means that the damage zone is infinite in this direction and that the solution is, in effect, one dimensional. Substitution of these solutions in the constitutive relations (20) and then in the equation of equilibrium (21) provides a linear algebraic homogeneous system of equation:

$$[\mathbf{n}^* L \mathbf{n}^*] \cdot A = 0$$

$$\text{where} \quad \mathbf{n}^* = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \text{ in the } (x, x_2) \text{ coordinate system} \tag{23}$$

L is the tangent stiffness of the material. Note the above analysis follows exactly the steps detailed in Pijaudier-Cabot and Benallal (1993), except that the orientation of the localization band determined by $\mathbf{n}^*$ is here fixed instead of being unknown. This is a simplifying assumption which is motivated from the definition of the equivalent tensile

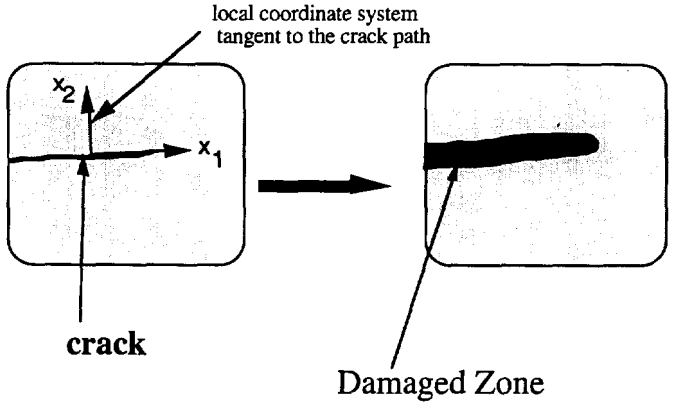


Fig. 1. Damage zone equivalent to a crack.

strain whose expression in a rate form may be difficult to obtain in a close form for any arbitrary perturbation (the function which defines the positive part of a scalar is not derivable in all instances). The orientation of the damage localization band is fixed on the basis of finite element results on tensile specimens (Saouridis, 1988).

In order to admit a non trivial solution, the determinant of the homogeneous system in eqn (23) must vanish. This condition provides the angular frequency which is also a function of the initial state of strain and damage:

$$\frac{(1-D^0)}{\varepsilon_{22}^0 \left.\dfrac{\partial f}{\partial \bar{\varepsilon}}\right|_{\varepsilon}} = \bar{\alpha}(\omega, l_c) \tag{24}$$

where $\bar{\alpha}(\omega, l_c)$ is the Fourier transform of the weight function which depends on the internal length l_c.

The calculation of the approximated fracture energy is based on the assumption that at the onset of strain localization, i.e., at the onset of localized cracking, the distribution of strain and damage jumps suddenly from a homogeneous distribution to a harmonic solution with the smallest possible wave length. This is based on stability considerations which require that the width of the localization zone should be as small as possible and on the assumption that the variation of angular frequency during intermediate states where maximum damage is neither equal to zero nor to 1 has a relatively small influence on the energy dissipation. In an infinite body, localization occurs suddenly and maximum damage jumps very rapidly to one. Therefore this assumption does not seem to be too far away from the exact process of localization in which the region in which damage evolves shrinks with increasing damage as the loading progresses. If a maximum strain ε_{rup} at which complete failure occurs is defined in the model, the maximum angular frequency within the interval $[0, \varepsilon_{rup}]$ is searched.

The angular frequency corresponding to the minimum possible value of the width of the damage zone along the axis x_2 of the local co-ordinate system is selected for deriving the approximate distribution of strain. The distribution of damage is obtained by substitution of the harmonic solution in the equations of evolution of damage (13) expressed in a rate form for uniaxial tension which are:

$$\dot{D} = \left[\frac{\varepsilon_{D0}(1-A_t)}{\bar{\varepsilon}^2} - \frac{A_t B_t}{\exp(B_t(\bar{\varepsilon}-\varepsilon_{D0}))}\right]\dot{\bar{\varepsilon}}$$

$$\dot{\bar{\varepsilon}}(x) = \frac{1}{V_r(x)} \int_r \alpha(s-x)\dot{\varepsilon}_{22}^0 \, \mathrm{d}s. \tag{25}$$

After substitution of the harmonic solution (22), integration of the equation of evolution of damage (25), between 0 and 1, and rescaling so that damage is equal to 1 along the crack path, we obtain:

$$D(x_2) = \frac{\displaystyle\int_{-\infty}^{+\infty} \alpha(x_2 - s_2)\eta(s_2)\, \mathrm{d}s_2}{\displaystyle\int_{-\infty}^{+\infty} \alpha(s_2)\eta(s_2)\, \mathrm{d}s_2}$$

with

$$\eta(x_2) = \cos(\omega_{max} x_2) \text{ if } x_2 \in \left[\frac{-\pi}{2\omega_{max}}, \frac{\pi}{2\omega_{max}}\right] \text{ and } D(x_2) \geqslant 0$$

$$\eta(x_2) = 0 \text{ elsewhere.} \tag{26}$$

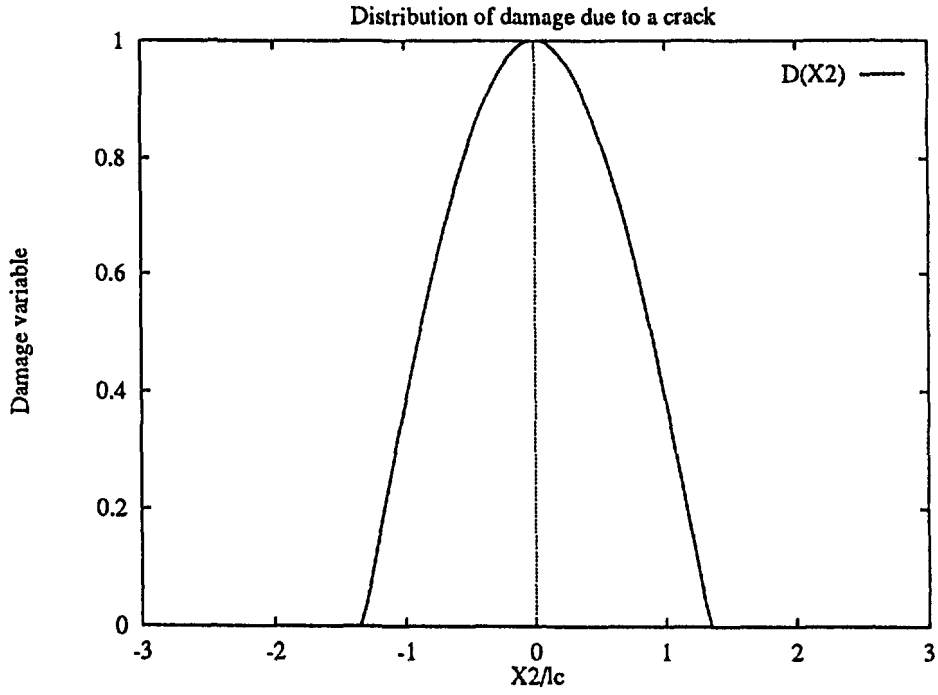


Fig. 2. Distributions of damage equivalent to a crack (variation along the local axis x_2).

The integration in eqn (26) is now only carried out according to co-ordinate x_2 because all the field quantities are constant with respect to the other co-ordinate, x_1, and because the non local average is normalized.

Figure 2 shows a typical distribution of damage obtained with the present approximation. This distribution depends in particular on the internal length l_c of the material. It is independent from the local coordinate x_1. The distribution of damage is constant along lines parallel to the crack. The damage variable is exactly equal to one on the line where the crack is located. The rest of the distribution of damage represents the fracture process zone, which is not collapsed on the crack line, as in fracture mechanics-based models.

The energy consumption due to crack propagation is the integral of the energy dissipation at each material point of co-ordinate x_2 in the fracture process zone which encountered damage up to $D(x_2)$. For a unit length of the localization band into an infinite body it comes from eqns (9), (17) and (25):

$$G_f = \int_{-\infty}^{+\infty} \left\{ \int_0^{\bar{\varepsilon}(x_2)} \frac{1}{2} E \varepsilon_{22}^2 \frac{\partial f}{\partial \bar{\varepsilon}} \mathrm{d}\bar{\varepsilon} \right\} \mathrm{d}x_2$$

$$\text{with} \quad \bar{\varepsilon} = \varepsilon_{22} \text{ (uniaxial tension).} \tag{27}$$

According to the equivalence set in Section 2.3, this dissipation energy corresponds to the energy released in fracture mechanics. Coming from infinite geometry this result represents the fracture energy for a large specimen and can be compared to the G_C deduced from the size effect method (Bazant, 1984).

Note that the width of the fracture process zone is implicitly fixed by eqn (26). It should be of the order of $2l_c$ since (i) this ratio was used for the calculation of the internal length obtained as $l_c \approx 3d_a$ where d_a is the maximum aggregate size, and (ii) this ratio was confirmed by numerical calculations. This result can serve as a check for the approximation method.

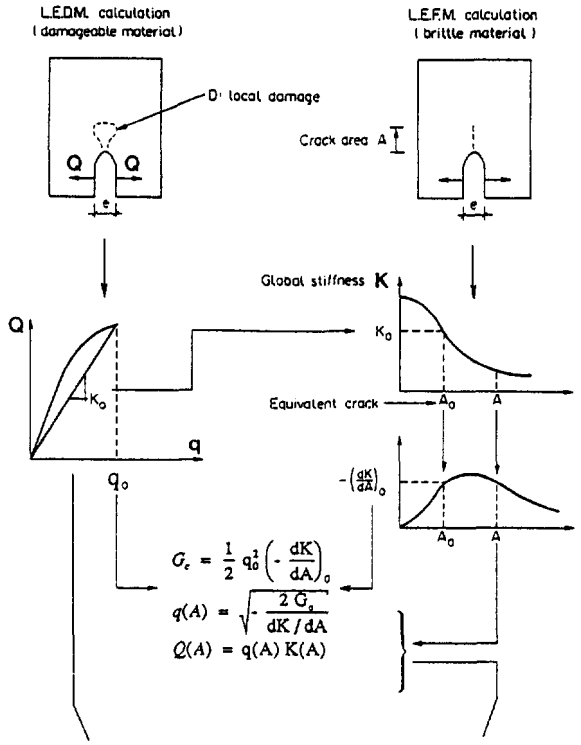


Fig. 3. Principle of the passage from damage mechanics to fracture mechanics.

4. APPLICATIONS

4.1. *Behaviour of a structure using a combined approach damage-fracture mechanics*

The structure considered is a compact tension specimen (Fig. 4a) a series of which was tested at LMT Cachan (Legendre, 1984). During the tests a lot of observations and measurements were performed showing that the global behaviour includes three stages (see Fig. 4c):

—OA, linear elastic,
—AB, damage with micro cracks, but without macro crack,
—BC, combination of both micro cracking and macro crack.

In order to describe this behaviour we propose two kinds of calculations:

—from O to B with a non local damage model,
—from B to C with linear elastic fracture mechanics.

The bridge from the first calculation to the other uses directly the equivalent crack concept presented in Section 3.1. This requires us to determine the evolution $K = K(A)$ and the derivative dK/dA as shown in Fig. 3. The global behaviour at point B (Q_0, q_0) gives the stiffness K_0 from which it is easy to deduce the corresponding equivalent crack A_0, the value of $(-dK/dA)_0$ and, finally, from (19), the critical value G_C. From G_C and the predetermined evolution of $(-dK/dA)$, Q and q can be computed for each new value of A, which gives the continuation of the Q-q curve. A summary of the whole procedure is given in Fig. 3. In the present case the following parameters have been used:

—non local damage calculation: $E = 34{,}500$ MPa, $\varepsilon_{D0} = 1.23$ E-04, $A_t = 0.8$, $B_t = 20{,}000$, $l_c = 30$ mm;
—critical fracture energy at point B: $Q_B = 18.9$ kN, $q_B = 0.2$ E-03 m, $K_B = 9.5$ E+04 kN/m $(-dK/dA)_B = 51$ E+05 kN/m^3, $G_C = 1/2 q_B^2 (-dK/dA)_B = 102$ N/m;
—LEFM calculation, from eqns (9), (13), one can deduce $q = \sqrt{(2G_C)/(-dK/dA)}$, from which it comes $Q = Kq$.

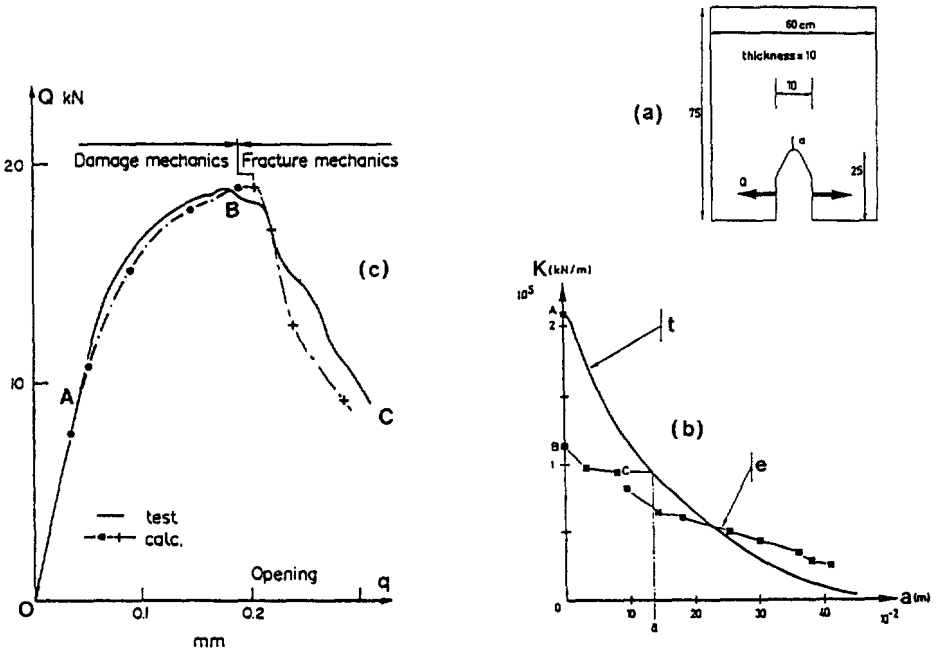


Fig. 4. Compact tension specimen : a—geometry (values are cm) ; b—evolution of the stiffness with the crack, (t) theoretical, (e) experimental ; c—global behaviour, calculation is performed using the bridge from damage to fracture.

From Fig. 4b it can be pointed out that :

—the equivalent crack length at point B is $a = 13$ cm,

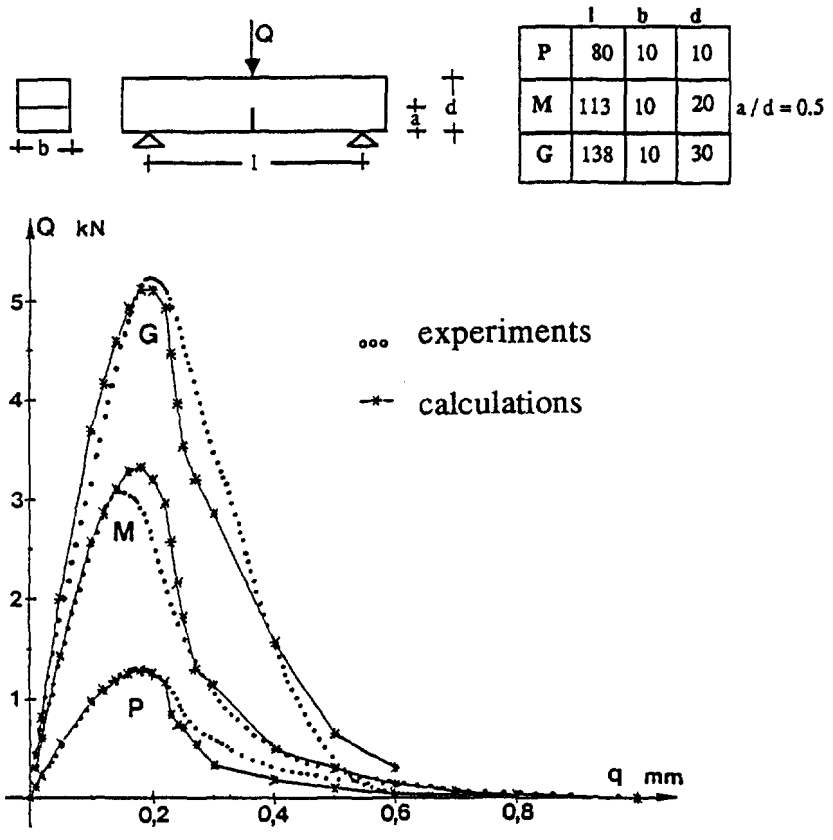
—the experimental curve which gives the evolution of the ratio $Q/q(\#K)$ vs the crack length measured directly on the surface (curve e) of the specimen; this is very different from the theoretical one (curve t). It confirms that the real crack appears close to the maximum of the load after a first decrease of the stiffness due to micro-cracking, and that the evolution on the surface is faster than inside the specimen, which entails that the remaining strength decreases softly, this result has been shown by other authors (Bascoul *et al.*, 1987). One of the conclusions is that it is not easy to deduce fracture parameters directly from crack lengths measured on the surface.

The part of the behaviour deduced from LEFM calculation looks in accordance with the experimental curve. However, the softening slope seems greater than in the experimental curve, this indicates that G_C increases during the evolution. On the other hand, we may notice that the value determined from the analytical calculation, Section 3.2, is $G_f = 115$ N/m. It is close but greater than G_C coming from the equivalent crack concept.

4.2. *Prevision of G_c with damage mechanics for large specimen from size effect*

The structures considered for this presentation are notched beams of different sizes, the geometry of which is given in Fig. 5. The tests have been done at Lund University (Horvath and Persson, 1984) and the calculations, using a non local damage model, have been performed by Saouridis (1988). Deduced from other experiments and from the smallest beam (P), the parameters introduced into the damage calculation have been used for predicting the responses of the specimens of the two other sizes. They are $E = 32{,}300$ MPa, $\varepsilon_{D0} = 3.0$ E-05, $A_t = 0.6$, $B_t = 3700$, $l_c = 30$ mm.

Figure 5 shows the good agreement between experimental and computed load-deflexion curves, as well for the peak as for the post peak. In order to explore the size effect, we used the classical presentation into the log-log stress-size diagram (Fig. 6.) From these results and using a regression line, the Bazant size effect law has been determined :



	l	b	d
P	80	10	10
M	113	10	20
G	138	10	30

a/d = 0.5

Fig. 5. Notched beams: a—geometry; b—computed and experimental load-deflexion curves.

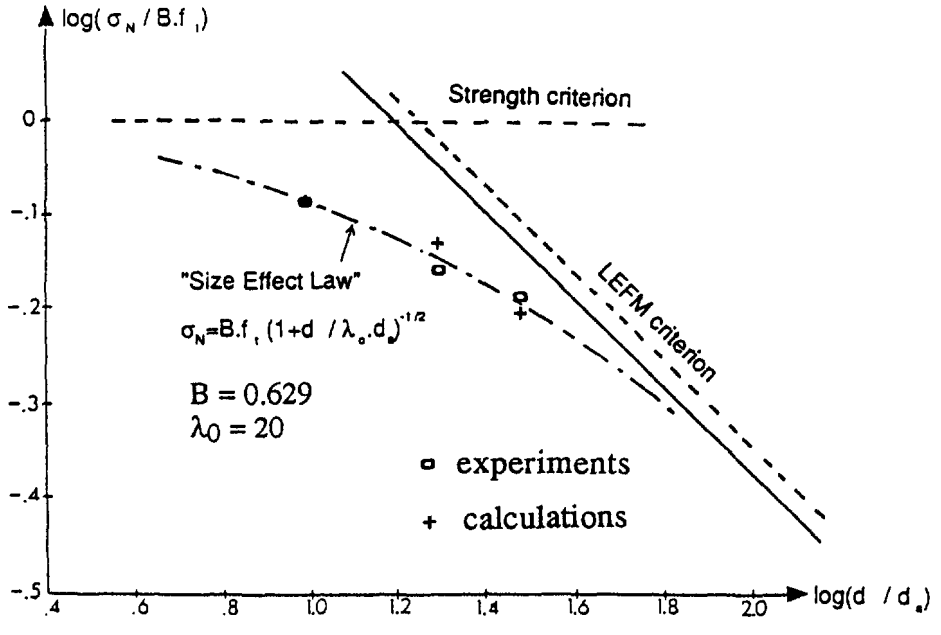


Fig. 6. Size effect on notched beams, determination of G_c for large specimen: (- - -) asymptote issued from the size effect law; (—) asymptote issued from analytical calculation of G_c.

$$\sigma_N = Bf_t(1 + d/\lambda_0 d_a)^{-1/2} \tag{28}$$

with $f_t = 30$ MPa, $d_a = 10$ mm, it was found $B = 0.629$ and $\lambda_0 = 20$.

The size effect method, which consists of the determination of fracture parameters from asymptotic values of the size effect law, leads to $K_C = 1.49$ MPa$\sqrt{\text{m}}$. With $G_C = K_C^2 (1-\nu^2)/E$ (plane strain conditions are used, because of the relative large thickness of the beam specimens), we obtain $G_C = 65.8$ N/m. From the non local parameters, the analytical determination of the fracture energy previously presented leads to $G_C = 55.6$ N/m or $K_C = 1.37$ MPa$\sqrt{\text{m}}$. Due to the assumption of a uniaxial tension state in the cracked zone, as indicated in Section 3.2, this approach is valid for a large specimen, the value obtained for K_C can be plotted into the log-log stress-size diagram as an asymptote, which is, *in fine*, close to the one found from the size effect law. Therefore, it is possible to obtain the fracture energy as a result of a computation with damage mechanics with a sufficient accuracy. This computation can be used to check the values of the parameters in the damage model.

4.3. *Behaviour of specimens with initial damage*

Conversely, it is possible to represent cracking by an equivalent damage zone. This equivalence follows from the same assumptions as those used for the derivation of the fracture energy. We consider that existing cracks have been created following a mode I process. Given a crack observed on a structure, the approximation given in (26) provides an equivalent map of damage. At the crack tip, the distribution of damage is lacking in the approximation. It is assumed to be circular and variable x_2 is replaced by the radius r defining the distance from the considered point to the tip of the crack. This map can be transferred on a finite element discretization and subsequent calculations aimed at evaluating the behaviour of the damaged structure can be performed. This method possesses the advantage of being completely disconnected from the mesh generation. An existing crack in the structure is modelled as a set of broken lines imputed interactively on the computer screen which shows the structure. Then, the distribution of damage is computed according to eqn (26) at each Gauss point of the mesh. Detailed information is given in a report by Bodé *et al.* (1995) where the predictions of this approximation was checked against experimental data on initially cracked concrete and fibre reinforced concrete specimens. We show here an example of computation on a compact tension specimen similar to that tested by Legendre (1984) shown in Fig. 4a with different material properties. The model parameters for this computation are:

$$\varepsilon_{D0} = 0.85\,10^{-4}, \quad E = 32{,}000\,\text{MPa}, \quad \nu = 0.2, \quad A_t = 1, \quad B_t = 4300,$$

$$l_c = 3.6\,\text{cm}, \quad \frac{2\pi}{\omega_{max}} = 3.77 l_c, \quad \varepsilon_{rup} = 3.6\,10^{-3}.$$

As a check of the model parameters, the fracture energy computed as the sum of the energy dissipation over the damage zone corresponds to usual values for this concrete, $G_C = 81$ N/m.

During loading (which is displacement controlled), a mode I crack propagates in the middle of the plate. In the computation, the position of the existing crack is first defined independently from the mesh (Fig. 7a). In the present case, the crack length is 10 cm. Once the crack location has been defined on the finite element mesh, a pre-processor computes the equivalent map of damage (Fig. 7b). Finite element calculation for any boundary conditions can be performed afterwards.

Figure 7c shows the response of the damaged compact tension specimen, i.e., containing a crack, compared to that of the undamaged specimen. The load reduction factor, defined as the ratio of the maximum carrying capacity of the cracked plate to the carrying capacity on the uncracked plate is 0.7. Bodé *et al.* (1995) have conducted more extensive comparisons on this particular test and on fibre reinforced concrete beams which revealed that the average accuracy of this approximate method for determining the load carrying capacity of damaged structures is about 20%.

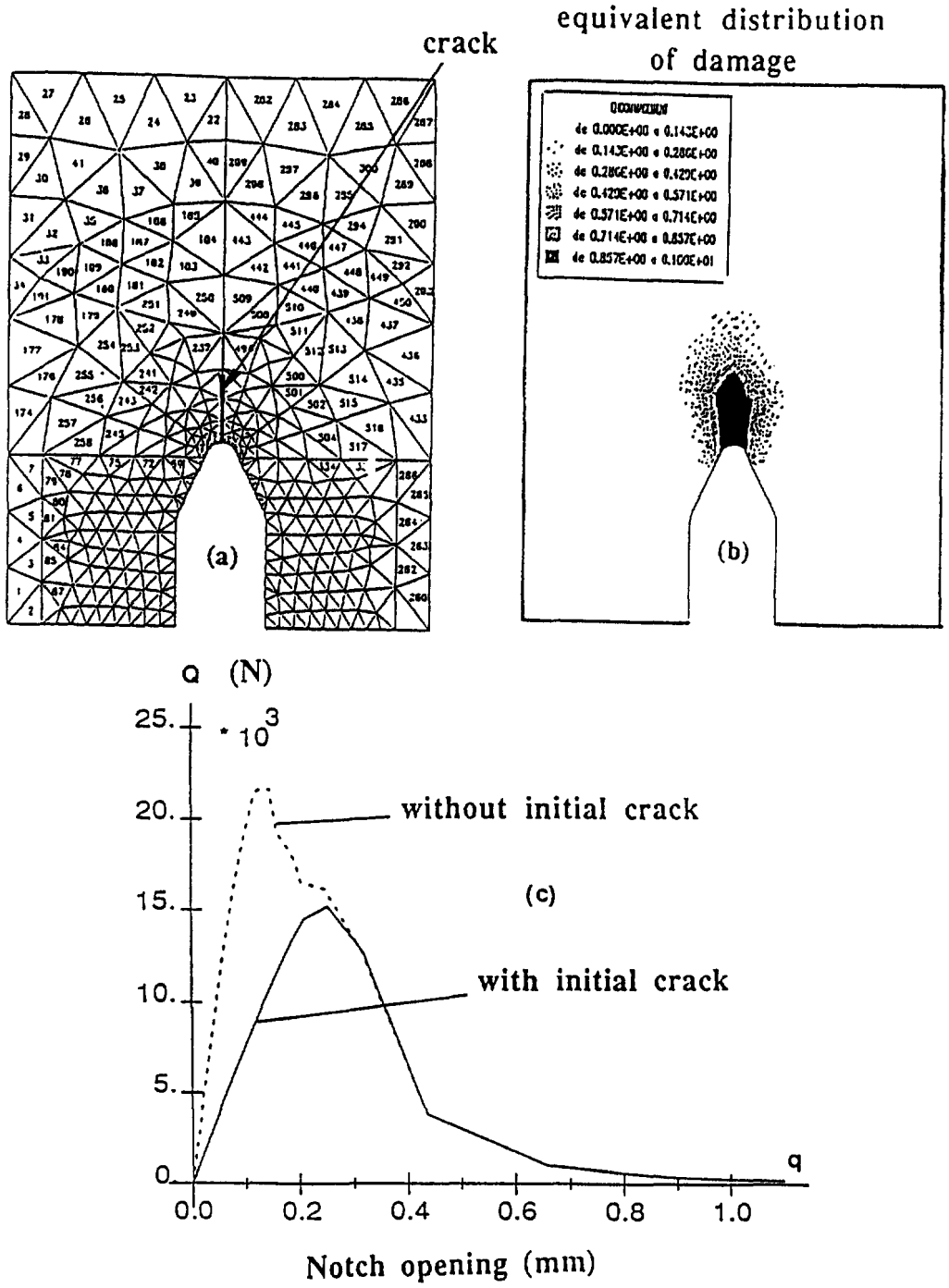


Fig. 7. Analysis of the behaviour of the compact tension test using the bridge from fracture to damage; a—initial crack; b—equivalent distribution of damage; c—simulation of the response of the cracked plate compared with the virgin plate.

5. CONCLUSIONS

Fracture mechanics and damage mechanics are two correlated theories. Damage mechanics is useful for the prevision of the onset of cracking in large specimens, fracture mechanics can be implemented for modelling the subsequent crack propagation. Non local damage models encompass both the prediction of crack initiation and propagation. From thermodynamics considerations, relationships between the fracture and non local damage theories have been shown in this paper.

There are two major uses of the proposed equivalence between fracture and damage:

(i) Damage and fracture mechanics combination:

When non local damage parameters are known, a first stage of calculation can be performed to obtain the zone where damage localizes (onset of cracking). From these results, an equivalent crack and the corresponding fracture energy can be deduced. This value presents the advantage to be the best adequate value for the situation considered, even if linear fracture mechanics is not perfectly respected. In the second stage, linear elastic fracture mechanics can be used. This is, obviously, more efficient if the crack path is previously known because the finite element mesh can be properly designed for this. For large structures, the computing time is by far reduced compared to a computation with a non local damage model because the continuum approach requires a larger number of finite elements in order to discretise the fracture process zone where the gradients of damage and strain are very high.

Conversely, it is possible to represent cracking by an equivalent damage zone. This equivalence follows from the same assumptions as those used for the derivation of the fracture energy from the damage approach. Given a crack observed on a structure, the approximation yields an equivalent map of damage. This map can be transferred on a finite element discretization and subsequent calculations aimed at evaluating the response of the damaged structure can be performed. This strategy may be used in order to investigate the residual safety of any structure for which subsequent cracking is not known.

(ii) Identification of fracture energy parameters for large scale specimens:

It has been shown that it was possible to compute the fracture energy from the damage model. Due to the size of the process zone which is stabilised when large scale structures are considered, the critical value for large specimens can be determined directly. The value of the fracture energy for large specimens can also be deduced from size effect experiments. Any damage model consistent with the linear elastic fracture approach which holds in the limit of specimens of infinite size should predict the correct fracture energy. This can be very useful, either to check the identification of the parameters in the damage model, or to determine them knowing the value of the fracture energy.

REFERENCES

Bascoul, A., Kharchi, F. and Maso, J. C. (1987). Concerning the measurement of the fracture energy of a micro concrete according to the crack growth in a three points bending test on notched beams. In *Proc. SEM-RILEM Int. Conf. Fracture of Concrete and Rock* (Edited by S. P. Shah and S. E. Swartz), Houston: Society of Experimental Mechanics, Bethal, CT.

Bazant, Z. P. (1976). Instability, ductility and size effect in strain-softening concrete. *ASCE J. Engng Mech.* **102,** 331–344.

Bazant, Z. P. and Oh, B. (1983). Crack band theory for fracture of concrete. *RILEM Materials Struct.* **16,** 155–177.

Bazant, Z. P. (1984). Size effect in blunt fracture; concrete, rock, metal. *J. Engng Mech., ASCE* **110,** 518–535.

Bazant, Z. P. (1986). Mechanics of distributed cracking. *Appl. Mech. Rev., ASME* **39,** 675–705.

Bazant, Z. P. and Pijaudier-Cabot, G. (1989). Measurement of characteristic length of non local continuum. *J. Engng Mech., ASCE* **115,** 755–767.

Bazant, Z. P. and Ozbolt, J. (1990). Nonlocal microplane model for fracture damage and size effect in structures. *ASCE J. Engng Mech.* **116,** 2485–2505.

Bodé, L., Tailhan, J. L., Pijaudier-Cabot, G., La Borderie, Ch. and Clément, J. L. (1995). Failure analysis of initially cracked concrete structural components. Int. Report Laboratoire de Mécanique et Technologie, Cachan, France..

Carol, I., Bazant, Z. P. and Prat, P. C. (1991). Geometric damage tensor based on microplane model. *ASCE J. Engng Mech.* **117,** 2429–2448.

de Borst, R. and Nauta, P. (1985). Non-orthogonal cracks in a smeared finite element model. *Engng Computing* **2,** 35–46.

de Borst, R., Muhlhaus, H. B. and Pamin, J. (1992). A gradient dependent continuum model for mode I fracture in concrete and rock. In *Fracture Mechanics of Concrete Structures* (Edited by Z. P. Bazant), pp. 251–259. Elsevier Publishers, Oxford.

de Borst, R., Pamin, J., Pijaudier-Cabot, G., Bodé, L. and Bui, H. D. (1992). Constitutive relations and design rules for cementitious composites. Brite-Euram Project P. 3275: Failure mechanics of fibre reinforced concrete and pre-damaged structures, Task report, Laboratoire de Mécanique et Technologies, Cachan, France.

Elfgren, L. (Ed.) (1989). *Fracture Mechanics of Concrete Structures, from Theory to Applications.* Chapman and Hall, London.

Hillerborg, X., Modeer, M. and Petersson, P. E. (1976). Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cement and Concrete Res.* **6,** 773–782.

Horvath, R. and Persson, T. (1984). The influence of the specimen size on the fracture energy of concrete. *Report TVBM*-5005, Lund Institute of Technology, Lund, Sweden.
Kaplan, M. F. (1961). Crack propagation and the fracture of concrete. *A.C.I. J.* **58,** 591–610.
Legendre, D. (1984). Prévision de la ruine des structures en béton par une approche combinée: mécanique de l'endommagement—mécanique de la rupture. Thèse de Doctorat, Université Paris 6—LMT Cachan.
Lemaitre, J. and Chaboche, J. L. (1990). *Mechanics of Solid Materials*. Cambridge University Press, Cambridge.
Lemaitre, J. (1992). *A Course on Damage Mechanics*. Springer-Verlag, Berlin.
Mazars, J. (1977). Existence of a critical strain energy release rate for concrete. In *Proc. of I.C.F.* 4 (Edited by D. M. R. Taplin), pp. 1205–1209. University of Waterloo publishers, Waterloo, Canada.
Mazars, J. (1986). A description of micro and macro-scale damage of concrete structure. *J. Engng Fract. Mech.* **25,** 729–737.
Mazars, J. and Pijaudier-Cabot, G. (1989). Continuum damage theory—application to concrete. *J. Engng Mech. ASCE* **115,** 345–365.
Mazars, J., Pijaudier-Cabot, G. and Saouridis, C. (1991). Size effect and continuous damage in cementitious materials. *Int. J. Fract.* **51,** 159–173.
Pijaudier-Cabot, G. and Benallal, A. (1993). Strain localization and bifurcation in a non local continuum. *Int. J. Solids Structures* **30,** 1761–1775.
Planas, J., Elices, M. and Guinea, G. V. (1993). Cohesive cracks versus non local models—closing the gap. *Int. J. Fract.* **63,** 173–187.
Saouridis, C. (1988). Identification et numérisation objectives des comportements adoucissants: une approche multiéchelle de l'endommagement du béton. Thèse de Doctorat, Université Paris 6, LMT Cachan.
Saouridis, C. and Mazars, J. (1992). Prediction of the failure and size effect in concrete via a bi scale damage approach. *Engng Computations J.* **9,** 329–344.
Simo, J. C. (1989). Strain softening and dissipation: a unification of approaches. In *Cracking and Damage* (Edited by J. Mazars and Z. P. Bazant), pp. 440–461, Elsevier Science, Oxford.
Simo, J. C., Oliver, J. and Armero, F. (1993). An analysis of strong discontinuities induced by strain softening in rate independent inelastic solids. *Computational Mech.* **12,** 277–296.